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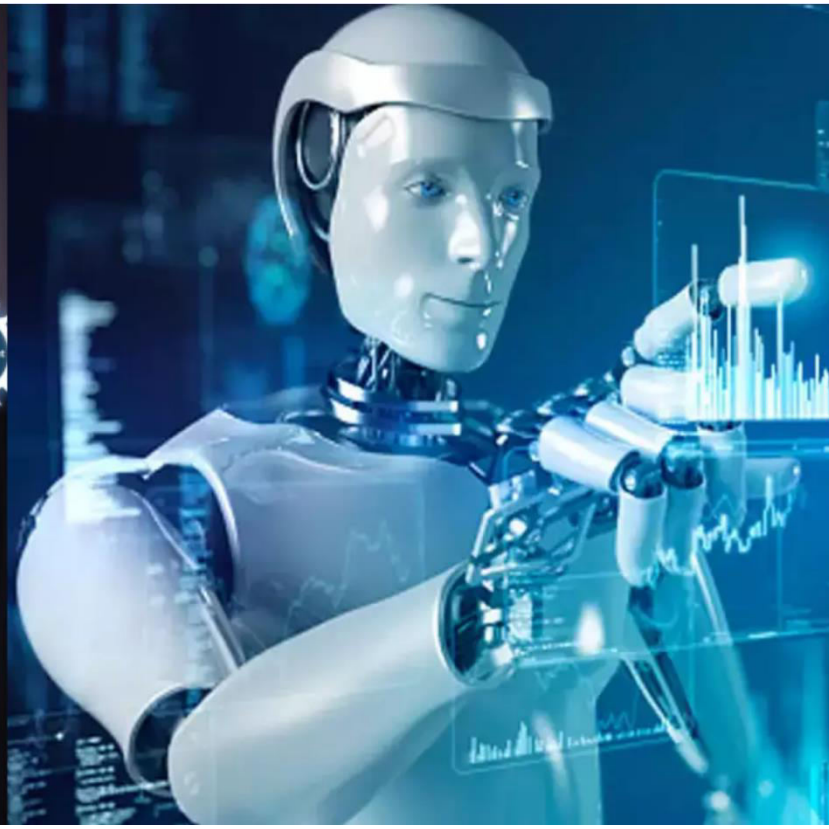

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Interrelation between Stability Conditions and Beta Invariant Substructures

D. Nethra Sharon[1], Dr. D. Radha[2]

Research Scholar, PG & Research Department of Mathematics, A.P.C. Mahalaxmi College for Women, Thoothukudi, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamil Nadu, India[1]

Associate Professor, PG & Research Department of Mathematics, A.P.C. Mahalaxmi College for Women, Thoothukudi, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamil Nadu, India[2]

Corresponding author: sharonnethra@gmail.com

ABSTRACT: The present study investigates the interrelation between stability conditions and invariant substructures in β–nearrings, emphasizing how variations in stability properties influence the internal algebraic organization of these systems. Nearrings, being noncommutative generalizations of rings, allow one-sided distributivity, leading to the emergence of rich and flexible substructures. Within this framework, β–nearrings extend classical concepts by introducing β–operations that preserve certain structural consistencies under homomorphic and compositional mappings. This paper explores the behavior of pseudo-stable**,** strong β–nearrings**,** and reverse pseudo-stable β–nearrings, analyzing their effect on invariant β–subgroups and ideal formations. The study establishes that in strong β–nearrings, every β–subgroup exhibits invariance under specific distributive operations, while pseudo-stable β–nearrings demonstrate idempotent stability, ensuring predictable algebraic outcomes. Furthermore, it is shown that zero symmetric β–nearrings with mate functions give rise to Boolean algebra structures over the set of β–subgroups, revealing a profound connection between stability, idempotency, and logical structure. The results contribute to a deeper understanding of how stability conditions govern substructure behavior in β–nearrings and open avenues for further exploration in the classification and application of near-ring-based algebraic systems.

KEYWORDS - β–nearrings , homomorphism, idempotent , Boolean, distributivity

I. INTRODUCTION

Near-rings are algebraic structures that extend and generalize the concept of rings by relaxing one or more of their defining axioms, particularly the distributive law. Formally, a near-ring (N, +, ·) is a nonempty set N equipped with two binary operations such that (N,+) forms an additive group (not necessarily abelian), and (N,·) forms a semigroup. The crucial distinction from rings is that multiplication in a near-ring is required to be only one-sided distributive either right or left distributive over addition. In this paper, N denotes a right near-ring**.** This single-sided distributivity allows a far greater degree of freedom than the ring structure, leading to the emergence of many novel algebraic behaviours that are absent in fully distributive systems. The study of near-rings arises naturally in various branches of mathematics and theoretical computer science. Near-rings appear in combinatorics, where they describe systems of functions under pointwise addition and composition; in automata theory, where transition mappings form near-ring structures; and in geometry, where certain incidence-preserving transformations can be modelled using near-ring operations. Furthermore, the endomorphism near-ring of a group—consisting of all self-maps of a group under pointwise addition and composition—provides one of the most fundamental examples illustrating the generality of the near-ring concept. Because a near-ring lacks full distributivity, its internal structure is much richer and more flexible than that of a ring. This asymmetry between left and right operations gives rise to a variety of interesting substructures, such as ideals, quasi-ideals, and bi-ideals, each exhibiting unique behaviours depending on the distributive side considered. These substructures play a crucial role in understanding how different algebraic properties, such as idempotency, stability, and commutativity, interact within the framework of near-rings.

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II. PRELIMINARIES

Definition 2.1
A right near ring is a non-empty set N together with two binary operations "+" and "·" such that
(i) $(N,+)$ is a group (not necessarily abelian)
(ii) $(N,\cdot)$ is a semigroup
(iii) $(n_1+n_2)n_3 = n_1n_3 + n_2n_3$ for all $n_1, n_2, n_3 \in N$

Definition 2.2
A near ring N is said to be zero symmetric if $N = N_0$ where $N_0 = \{n \in N / n0 = 0\}$

Definition 2.3
A subgroup M of the additive group $(N,+)$ is called a left N-subgroup of N if $MN \subseteq M$

Definition 2.4
A subgroup M of the additive group $(N,+)$ is called a N-subgroup of N if $NM \subseteq M$

Definition 2.5
A subgroup M of the additive group $(N,+)$ is called an invariant N-subgroup of N if $NM \subseteq M$ and $MN \subseteq M$

Notation 2.7

- E denotes the set of all idempotent of N (e in N is called an *idempotent* if $e^2 = e$).
- L denotes the set of all nilpotent of N (a in N is *nilpotent* if $a^k = 0$ for some positive integer k).
- $C(N) = \{n \in N / nx = xn \ \forall\ x\ in\ N\}$- centre of N.

Definition 2.8
A normal subgroup of I of $(N,+)$ is called *ideal* of N if
a) $IN \subseteq I$
b) b) $\forall\ n, n' \in N$ and $\forall\ i \in N$ such that $n(n'+i) - nn' \in I$
Normal subgroup R of $(N,+)$ with a) is called right ideals of N and normal subgroup R of $(N,+)$ is said to be left ideal.

III. MAIN RESULTS

Theorem 3.1
Let N be strong B_1 near ring. If N is β_4 near ring then $Nax \subseteq Na \cap Nx$ for all $a, x \in N$.
Proof
Let $y \in Nax$ for all $a, x \in N$
For some $n \in N$, $y = nax = (na)x \in Nx$
$\Longrightarrow y \in Nx$ ⇢ (1)
Now again let $y \in Nax$ for all $a, x \in N$
Since N is strong B_1 near ring, $y \in Nxa$ for all $a, x \in N$
For some $m \in N$, $y = mxa = (mx)a \in Na$
$\Longrightarrow y \in Na$ ⇢ (2)
From equations (1) and (2) it follows $y \in Na \cap Nx$
Therefore, $Nax \subseteq Na \cap Nx$ for all $a, x \in N$.

Theorem 3.2
Let N be a β_4 near ring with identity 1. Then
a) N is zero symmetric b) Every N-subgroup of N is invariant.

Proof
a) Since N is β_4 near ring for all $x, y \in N$, $xNy = Nyx$

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Put $y = 1$, $x.N.1 = N.1.x \Longrightarrow xN = Nx$ ($\because N$ has identity 1)
Now when $x = 0, 0N = N0 = 0$ ($\because 0N = 0$)
Therefore $N0 = 0$,which proves N is zero symmetric.

b) Let M be a N-subgroup of N.By definition, $NM \subseteq M$ ⇢ (1)
Also, we know that, $M = \sum_{x\in M} Nx$ ⇢ (2)
Consider $NxN = N(xN.1)$ ($\because N$ has identity 1)
$= NN.1.x$ ($\because$ N is β_4 near ring)
$= NNx \subseteq Nx$ ⇢ (3)
Now $MN = (\sum_{x\in M} Nx)N$ [By equation (2)]
$\subseteq \sum_{x\in M}(NxN)$
$\subseteq \sum_{x\in M}$ Nx [By equation (3)]
$= M$ [By equation (2)]
Therefore, $MN \subseteq M$ ⇢ (4)
From equations (1) and (4) we conclude every N-subgroup of N is invariant.

Theorem 3.3
Let N be a pseudo-stable β_4 near ring with idempotent then it satisfies $xNx = Nx$

Proof
Since N is a β_4 near ring we have,
$xNy = Nyx$
$xNx = Nxx$ since N is pseudo-stable
$= Nx$ since N has idempontency
Therefore $xNx = Nx$.

Theorem 3.4
Let N be β_4 near ring with idempotent element $e \in E$and is commutative then it is stable.

Proof
Let N be a β_4 near ring then,
$xNy = Nyx$
For $x = y = e$, we have
$\Rightarrow eNe = Nee = Ne = eN$ (since $e \in E$ and commutative)

Therefore, satisfies stableness.

Theorem 3.5
Let N be a zero symmetric β_4 near ring with mate function f.Then the set J of all N-subgroup is a Boolean Algebra under the usual set inclusion.

Proof
Let J be set of all N-subgroup (i.e.) $J = \{Nx/\ x \in N\}$.Since N is zero symmetric β_4 near ring with mate function f , we have $E \subseteq C(N)$ and let N be abelian.
Also $Nx \cap Ny = Nxy$ and $Nx + Ny = Nz$ for some $z \in N$
$\Longrightarrow J$ is lattice under usual set inclusion.
Since N has mate function f. For every $x, y, z \in N$ $x = xf(x)x$, $y = yf(x)y$, $z = zf(z)z$ for some $f(x), f(y), f(z) \in N$ and $f(x)x, f(y)y, f(z)z \in E$. Also $Nx = Nf(x)x$, $Ny = Nf(y)y, Nz = Nf(z)z$.
Consider $(f(x)xf(y)y)^2 = (f(x)xf(y)y)(f(x)xf(y)y)$
$= f(x)x(f(y)yf(x)x)f(y)y)$
$= f(x)xf(x)xf(y)yf(y)y$
$= (f(x)x)^2(f(y)y)^2$
$= f(x)xf(y)y$
$\Longrightarrow f(x)xf(y)y \in E.$

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Similarly, $f(x)xf(z)z \in E$.
Now we show that J is a distributive lattice. For all $nx, Ny, Nz \in J$ we have

$$\begin{aligned} Nx \cap (Ny + Nz) &= Nf(x)x \cap (Nf(y)y + Nf(z)z \\ &= Nf(x)x \cap N(f(y)y + f(z)z - f(y)yf(z)z) \\ &= Nf(x)x(f(y)y + f(z)z - f(y)yf(z)z) \\ &= N(f(x)xf(y)y + f(x)xf(z)z - f(x)xf(y)yf(z)z) \\ &= N(f(x)xf(y)y + f(x)xf(z)z - (f(x)x)^2 f(y)yf(z)z) \\ &= N(f(x)xf(y)y + f(x)xf(z)z - f(x)xf(y)yf(x)xf(z)z) \\ &= N(f(x)xf(y)y) + N(f(x)xf(z)z) \\ &= Nf(x)x \cap Nf(y)y) + Nf(x)x \cap Nf(z)z \\ &= Nx \cap Ny + Nx \cap Nz \end{aligned}$$

$\Longrightarrow J$ is a distributive lattice.
We claim that, if $Nx \subseteq Ny \subseteq Nz$ then there exist $w \in N$ such that $Ny \cap Nw = Nx$ and $Ny + Nw = Nz$.
Let $Nx \subseteq Ny \subseteq Nz \Longrightarrow Nf(x)x \subseteq Nf(y)y \subseteq Nf(z)z$.Then there exist $n_1, n_2 \in N$ such that $f(x)x = n_1 f(y)y = n_2 f(z)z$
Now $f(x)xf(y)y = (n_1 f(y)y)f(y)y = n_1(f(y)y)^2 = n_1 f(y)y = f(x)x$
Similarly, $f(x)xf(z)z = f(x)x$
Now $f(y)yf(x)x = f(y)y(n_1 f(y)y) = (n_1 f(y)y)f(y)y = n_1 f(y)y = f(x)x$
Similarly, $f(z)zf(x)x = f(x)x$
We get, $f(x)xf(y)y = f(y)yf(x)x = f(x)xf(z)z = f(z)zf(x)x = f(x)x$
Similarly, since $Ny \subseteq Nz \Longrightarrow Nf(y)y \subseteq Nf(z)z$.Then there exist $n_3 \in N$ such that $f(y)y = n_3 f(z)z$
Consider $f(y)yf(z)z = (n_3 f(z)z)f(z)z = n_3 f(z)z = f(y)y$
And $f(z)zf(y)y = f(z)z(n_3 f(z)z) = (n_3 f(z)z)f(z)z = n_3 f(z)z = f(y)y$
Again, we get $f(y)yf(z)z = f(z)zf(y)y = f(y)y$
Let $w = f(x)x + f(z)z - f(y)y$

$$\begin{aligned} w^2 &= (f(x)x + f(z)z - f(y)y)^2 \\ &= (f(x)x + f(z)z - f(y)y)(f(x)x + f(z)z - f(y)y) \\ &= f(x)x(f(x)x + f(z)z - f(y)y) + f(z)z(f(x)x + f(z)z - f(y)y) - f(y)y)(f(x)x + f(z)z - f(y)y) \\ &= f(x)xf(x)x + f(x)xf(z)z - f(x)xf(y)y + f(z)zf(x)x + f(z)zf(z)z - f(z)zf(y)y - (f(y)yf(x)x + f(y)yf(z)z - f(y)yf(y)y) \\ &= f(x)x + f(x)x - f(x)x + f(x)x + f(z)z - f(y)y - f(x)x + f(y)y - f(y)y \\ &= f(x)x + f(z)z - f(y)y = w \end{aligned}$$

Now we can say w is an idempotent element.
Consider $f(y)yw = wf(y)y$

$$\begin{aligned} &= (f(x)x + f(z)z - f(y)y)(f(y)y) \\ &= (f(x)xf(y)y + f(z)zf(y)y - f(y)yf(y)y) \\ &= f(x)x + f(y)y - f(y)y = f(x)x \end{aligned}$$

Therefore, $Nf(y)y \cap Nw = Nf(y)yw = Nf(x)x \Longrightarrow Ny \cap Nw = Nx$
Consider $Nf(y)y + Nw = N(f(y)y + w - f(y)yw)$

$$\begin{aligned} &= N(f(y)y + w - f(x)x) = Nf(z)z = Nf(z)z \\ &\Longrightarrow Ny + Nw = Nz \end{aligned}$$

From above observations it is clear that J is a Boolean Algebra.

Theorem 3.6
Let N be a β_4 near ring that is reverse pseudo stable and I be completely prime ideal of N.If $aNb \subseteq I$, then $a \in I$ or $b \in I$.

Proof
Let $aNb \subseteq I$
By definition of β_4 near ring we have for every $a, b \in N$, $aNb = Nba$
$\Rightarrow Nba \subseteq I$
Then for every $n \in N, nba \in I$

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Since completely prime, $(nb)a \in I \Rightarrow nb \in I$ or $a \in I$
Case i)
If $a \in I$ .Then the proof is done.
Case ii)
If $a \notin I$ then $nb \in I$
$\Rightarrow Nb \subseteq I$
For any $b \in N$,we get $bb = b^2 \in I$
By complete primeness, $b \in I$
Therefore, we conclude, $aNb \subseteq I \Rightarrow a \in I$ or $b \in I$.

Theorem 3.7
Anti-homomorphism image of β_4 near ring is β_1 near ring

Proof
Let $\varphi: N \to M$ be anti-homomorphism where N is β_4 near ring
Consider $\varphi(xNy) = \varphi(x)\varphi(N)\varphi(y)$
$= \varphi(y)\varphi(N)\varphi(x)$ (since anti-homomorphism)
$= \varphi(yNx)$
$= \varphi(Nxy)$ (since β_4 near ring)
By definition of β_1 near ring ,
we get M as β_1 near ring.

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